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RESEARCH NOTE

Propagators and Feynman diagrams for laterally heterogeneous elastic media

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SUMMARY

The propagator for coupled-mode elastic waves can be cast into a number of different representations, which emphasize particular aspects of the wave propagation in a laterally heterogeneous medium. One representation has the form of a generalized scattering operator and contains a quantity that can be interpreted as the lateral impedance. Another representation reduces naturally to the JWKB approximation for smoothly varying media with no mode coupling. The propagator solution for the fields in a laterally heterogeneous elastic medium with weak random boundary fluctuations leads naturally to the application of Feynman diagram techniques for the derivation of Dyson's equation and the Bethe–Salpeter equation for the propagator mean and covariance, respectively. The diagram techniques are reviewed and their utility for solution of random media elastic wave problems is demonstrated.

Key words: mode coupling, scattering, wave propagation.

1 INTRODUCTION

The Born approximation is a non-energy conserving, single-scattering approximation suitable from describing wave propagation in a medium containing a dilute distribution of weak scatterers. A signal propagating in a medium with a dense distribution of strong scatterers becomes incoherent as phase information is lost after travelling some distance large compared to the mean free path between scattering events. Propagation through this latter medium can effectively and efficiently be treated with radiative transport theory.

Between the very weak (Born approximation) and the very strongly (radiative transport theory) scattering regimes is the province of multiple-scattering theory, which takes into account a succession of scattering events, including multiply scattered waves.

Diagram methods, originally developed for computations in quantum field theory by Richard Feynman have a long history of application to classical problems as well. The first such classical application of diagram methods to classical physics problems may have been Foldy's (1945) paper treating sound propagation in a bubbly fluid. Tatarskii (1971) used diagram methods to model electromagnetic wave propagation in a turbulent atmosphere. More recently Ye & Ding (1995) and Henyey (1999) have used diagram methods to derive corrections to Foldy's (1945) original theory. This latter work has led to improved understanding of acoustic wave propagation in bubbly fluids.

In this article, we couple the classical propagator for elastic waves with diagram methods to describe modal scattering in a 2-D medium. We restrict our treatment to 2-D, as travelling wave modal methods in 3-D elastic are just beginning to be developed (Kennett 1998) and even with modern computers would be difficult to implement in their entirety. The ultimate goal is of course to be able to model propagation in a 3-D heterogeneous medium. The application of such techniques as described here is to propagation of bottom interacting acoustic waves, e.g. shallow-water, low-frequency acoustics and T waves, and to strongly scattered regional seismic waves.

The purpose of this article is two-fold. The first is to consider several different forms of the displacement–stress propagator for a 2-D elastic medium. These alternative forms emphasize different aspects of the propagation and may yield improved physical insight and enhanced computational efficiency for certain specific problems. The second purpose of this article is to describe Feynman diagram techniques useful for characterizing the propagation in 2-D elastic media with random fluctuations. These diagram techniques provide powerful tools for the computation of quantities of physical significance such as the mean elastic wave field and its covariance in a random elastic medium. Beginning with the propagator, the diagram techniques are used to derive both the Dyson equation for the propagator mean and the Bethe–Salpeter equation for the propagator covariance.

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Local coupled-mode methods are formally exact and, hence, can be employed for strongly heterogeneous problems. However, the strength of the heterogeneity that can be treated is ultimately limited by available computational resources. A very heterogeneous model and a broadband source could require a number of local mode sets exceeding currently available, or even near future, computer capabilities for their computation. Clearly the solution method for a particular problem must be matched to the complexity of the problem, precision required and available resources including the patience of the researcher.

The first application of the propagator to elastic waves was probably by Gilbert & Backus (1966), who derived the propagator for the first-order differential system describing wave propagation in a 1-D elastic medium. They also discussed the product integral representation of the propagator. The propagator for 1-D elastic media has become a standard representation for constructing the elastic wavefield solution and is covered in detail in standard books such as Aki & Richards (1980) and Kennett (1983). Beginning with the Gilbert & Backus (1966) product integral representation of the propagator, we construct three additional representations of the propagator, each of which emphasizes particular physical aspects of the propagation.

The starting point for the theory in this paper and Park & Odom (1999) is Maupin's (1988) local coupled-mode formulation of elastic wave propagation in a laterally heterogeneous medium. We assume the laterally heterogeneous medium to be bounded on the left and the right by laterally homogeneous (1-D varying in depth only) models, which do not necessarily have to be the same. Mathematically, the problem is a boundary value problem for propagation in the horizontal direction. The complete stress–displacement field $\mathbf{v}(x, z; \omega)$ in the laterally heterogeneous medium is represented as a superposition of local modes $\mathbf{v}^{\alpha}(z, k_{\alpha}; x)$ with position dependent expansion coefficients $c_{\alpha}(x)$

$$\mathbf{v}(x, z; \omega) = \sum_{\alpha} c_{\alpha}(x)\mathbf{v}^{\alpha}(z, k_{\alpha}; x)\mathrm{e}^{i\phi_{\alpha}(x)}, \tag{1}$$

where

$$\phi_{\alpha}(x) = \int_{x_s}^{x} k_{\alpha}(\xi)\, d\xi. \tag{2}$$

The range-dependent amplitudes $c_{\alpha}(x)$ are found by solving the evolution equation

$$\frac{\partial c_{\alpha}}{\partial x} = B_{\alpha\beta} c_{\beta}. \tag{3}$$

The matrix $B_{\alpha\beta}$ is the mode–mode coupling matrix. The evolution eq. (3) includes both forward and backward mode amplitudes, and the coupling between them. Direct integration of eq. (3) is not particularly stable. A more stable method is to convert eq. (3) to a Ricatti equation for the lateral transmission and reflection matrices. The non-linear Ricatti equation is then integrated backwards in space starting from the final conditions of unit transmission and no reflection from the laterally homogeneous medium bounding the laterally heterogeneous region on the right. The details of converting eq. (3) to a Ricatti equation and its integration for a simple laterally heterogeneous model can be found in Park & Odom (1998).

Note that our Fourier transform sign convention is the opposite to that used by Maupin (1988). With our sign convention a rightward propagating plane wave in a homogeneous medium has the phase term exp $i(kx - \omega t)$. Also a rapidly varying phase term has been removed from eq. (5). The slowly varying mode coupling matrix $B_{\alpha\beta}$ depends only on the difference in phase between mode α and mode β,

$$B_{\alpha\beta} = \hat{B}_{\alpha\beta} \exp(\phi_{\beta} - \phi_{\alpha}), \tag{4}$$

where $\hat{B}_{\alpha\beta}$ has no dependence on the phase. When propagation in a homogeneous, plane-layered medium is considered, there is no mode coupling, $B_{\alpha\beta} = 0$, and the amplitudes c_{α} are constants.

Employing the product integral representation, Park & Odom (1999) constructed the propagator for the horizontally travelling wave field in 2-D deterministic and random media. This was an extension of Maupin's (1988) deterministic theory. The starting points are the spatial evolution equations for the position-dependent mode amplitudes from elastic wave coupled-mode theory applied to a laterally heterogeneous medium with randomly rough interface boundaries. The random roughness is assumed to be a small perturbation to the layer boundaries of some mean reference structure. The geometry is described in the Appendix. Below are the spatial evolution equations (Park & Odom 1999) for the $\mathcal{O}(1)$ equation (eq. 5), which is equivalent to the evolution equation for a deterministic range-dependent medium, and $\mathcal{O}(\varepsilon)$ equation (eq. 6) for the coherent field and the scattered field written in matrix form:

$$\mathcal{O}(1): \frac{\partial \mathbf{c}^{(0)}}{\partial x} = \mathbf{B}\mathbf{c}^{(0)}, \tag{5}$$

$$\mathcal{O}(\varepsilon): \frac{\partial \mathbf{d}}{\partial x} = \mathbf{B}\mathbf{d} + (\mathbf{D} + \mathbf{E}\mathbf{B})\mathbf{c}. \tag{6}$$

Eq. (5) is identical to Maupin's (1988) evolution equation, eq. (3) above. Eq. (6) is discussed in Park & Odom (1999). The matrices **B**, **D** and **E** are $\nu \times \nu$ matrices of complex values, which are partitioned into terms for forward-to-forward, forward-to-backward, backward-to-forward and backward-to-backward coupling. The vectors $\mathbf{c}^{(0)}(x)$, $\mathbf{c}(x)$ and $\mathbf{d}(x)$, are the primary, coherent and scattered mode amplitudes, respectively; these are $\nu \times 1$ column vectors in the local mode space, where ν is the appropriate number of modes required to accurately model a signal. The vectors contain both forward and backward propagating mode amplitudes. The matrices **D** and **E** contain the contributions to the scattering from the stochastic part of the model. The primary mode amplitudes $\mathbf{c}^{(0)}(x)$ are the amplitudes that would exist in the laterally heterogeneous deterministic reference medium, i.e. the medium with no random heterogeneities. The coherent and scattered mode amplitudes, $\mathbf{c}(x)$ and $\mathbf{d}(x)$, respectively, are the amplitudes in the random medium. The coherent and scattered mode amplitudes are treated together, so an initially coherent signal decays with increasing propagation distance, and the scattered mode amplitudes will consequently increase. A statement of energy conservation in a non-attenuating medium is then

$$\left|\left|\mathbf{c}^{(0)}(x)\right|\right| = ||\mathbf{c}(x) + \mathbf{d}(x)||, \tag{7}$$

where $||\cdot||$ indicates the Euclidean norm.

2 THE COUPLED-MODE PROPAGATOR FOR THE DETERMINISTIC REFERENCE MEDIUM

Because we want to examine the complete wave propagator for the heterogeneous medium, we will first transform eq. (5) from slowly varying amplitudes back to the rapidly varying amplitudes, which include the local phase variation. For one case, which is clearly indicated below, it makes more sense to work with the slowly varying system. Transformation of eq. (5) is effected by substitution of

$$c_\alpha = \hat{c}_\alpha^{(0)} e^{-i\phi_\alpha(x)} \qquad \text{and} \qquad c_\beta = \hat{c}_\beta^{(0)} e^{-i\phi_\beta(x)} \tag{8}$$

into eq. (5). The resulting rapidly varying evolution equation is

$$\frac{\partial \hat{\mathbf{c}}^{(0)}}{\partial x} = i\mathbf{H}\hat{\mathbf{c}}^{(0)}, \tag{9}$$

where a new coupling matrix $\mathbf{H}$ is defined as

$$H_{\alpha\beta} \equiv (K_{\alpha\beta} + \hat{B}_{\alpha\beta}) \qquad \text{and} \qquad K_{\alpha\beta} \equiv k_\beta \delta_{\alpha\beta}, \tag{10}$$

i.e.

$$\mathbf{H} \equiv \begin{pmatrix} k_1 & \hat{B}_{12} & \cdots & \hat{B}_{1\nu} \\ \hat{B}_{21} & k_2 & \cdots & \hat{B}_{2\nu} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{B}_{\nu 1} & \hat{B}_{\nu 2} & \cdots & k_\nu \end{pmatrix}. \tag{11}$$

Here, k_α denotes the wavenumber of the αth mode. The vector $\hat{\mathbf{c}}^{(0)}$ is a vector of rapidly varying mode amplitudes.

The diagonal terms of the matrix $\mathbf{H}$ consist of the local wavenumbers and the off-diagonal terms are equal to the elements of the coupling matrix. The matrix $\mathbf{H}$ is a Hermitian matrix for the real eigenwavenumbers,

$$H_{\alpha\beta}^\dagger = (k_\beta\delta_{\alpha\beta} + \hat{B}_{\alpha\beta})^\dagger = k_\beta^*\delta_{\alpha\beta}^\dagger + \hat{B}_{\alpha\beta}^\dagger = k_\beta\delta_{\alpha\beta} + \hat{B}_{\alpha\beta} = H_{\alpha\beta}, \tag{12}$$

and the matrix $\hat{\mathbf{B}}$ is a function of the material properties, the slope and the local modes,

$$H_{\alpha\beta} = H_{\alpha\beta}\left(\rho, v_P, v_S, \dot{h}_n^0, \mathbf{v}^\alpha, \mathbf{v}^\beta\right). \tag{13}$$

The dot represents $\frac{\partial}{\partial x}$, ρ is the density, v_P is the P-wave speed, v_S is the S-wave speed, h_n^0 is the boundary function of the n^{th} reference boundary, and $\mathbf{v}^\alpha$ and $\mathbf{v}^\beta$ represent the local modes. The new transformed evolution eq. (9), therefore, becomes a differential equation for the rapidly varying amplitudes $\hat{c}^{(0)}(x)$. If the medium comprises homogeneous plane layers, the matrix $\hat{B}_{\alpha\beta} = 0$ and the solution to eq. (9) is easily seen to have the phase dependence $e^{\pm i k_\alpha x}$.

2.1 Evolution operator for an infinitesimal interval

We seek an evolution operator, i.e. propagator $\mathbf{U}$ which is defined via

$$\hat{\mathbf{c}}^{(0)}(x) = \mathbf{U}(x, x_0)\hat{\mathbf{c}}^{(0)}(x_0). \tag{14}$$

Now let us calculate the evolution operator for eq. (9) between two points separated by dx. To do this, write eq. (9) in the form (Cohen-Tannoudji *et al.* 1977)

$$\begin{aligned} d\hat{\mathbf{c}}^{(0)}(x) &= \hat{\mathbf{c}}^{(0)}(x + dx) - \hat{\mathbf{c}}^{(0)}(x) \\ &= i\mathbf{H}\hat{\mathbf{c}}^{(0)}\,dx, \end{aligned} \tag{15}$$

that is,

$$\hat{\mathbf{c}}^{(0)}(x + dx) = [\mathbf{I} + i\mathbf{H}(x)\,dx]\hat{\mathbf{c}}^{(0)}(x). \tag{16}$$

The evolution operator for the infinitesimal interval $[x, x + dx]$, the infinitesimal evolution operator, can be obtained from eq. (16) and the definition (14):

$$\mathbf{U}(x + dx, x) = \mathbf{I} + i\mathbf{H}(x)\,dx. \tag{17}$$

Because $\mathbf{H}(x)$ is Hermitian, $\mathbf{U}(x + dx, x)$ is unitary:

$$\begin{aligned} \mathbf{U}^\dagger\mathbf{U} &= (\mathbf{I} + i\mathbf{H}\,dx)^\dagger(\mathbf{I} + i\mathbf{H}\,dx) \\ &= \mathbf{I} + i(\mathbf{H} - \mathbf{H}^\dagger)\,dx + \mathcal{O}(dx^2) \\ &= \mathbf{I}. \end{aligned} \tag{18}$$

It follows that $\mathbf{U}(x, x_0)$ is also unitary because the finite interval $[x, x_0]$ can be divided into a very large number of infinitesimal intervals. A detailed proof is in Park & Odom (1999).

2.2 Definition and properties of product integral

The product integral $\breve{\int}$, which was first introduced to the elastic wave propagation problem by Gilbert & Backus (1966), is used to represent the propagator of the local mode for the laterally heterogeneous medium. As the solution of the differential equation

$$\frac{\partial \mathbf{P}(x, a)}{\partial x} = i\mathbf{H}(x)\mathbf{P}(x, a), \tag{19}$$

the product integral is defined as

$$\mathbf{P}(x, a) = \lim_{L\to\infty} \prod_{l=1}^{L} \exp\{i\mathbf{H}(\xi_l)\zeta_l\}$$
$$\equiv \breve{\int}_a^x \exp\{i\mathbf{H}(\xi)\}\, d\xi, \tag{20}$$

where the range interval (a, x) is divided into L parts, by introducing the mesh points $x_1 \le x_2 \le \cdots \le x_{L-1}$, $a \equiv x_0$ and $x \equiv x_L$. In the interval $x_{l-1} \le x \le x_l$, the intermediate point $\xi_l \equiv (x_{l-1} + x_l)/2$ and the length of the subinterval $\zeta_l \equiv x_l - x_{l-1}$, $l = 1, 2, \ldots, L$.

The product integral notation makes concise statements of rigorous results possible, which would be cumbersome otherwise. We introduce some principal properties of the product integral (DeWitt-Morette *et al.* 1979; Schulman 1981).

(i) Property 1:

$$\mathbf{P}(x, x) = \mathbf{I}, \tag{21}$$

$$\mathbf{P}(x, y) = \{\mathbf{P}(y, x)\}^{-1}. \tag{22}$$

(ii) Property 2:

$$\frac{\partial \mathbf{P}(x, y)}{\partial x} = i\mathbf{H}(x)\mathbf{P}(x, y), \qquad \frac{\partial \mathbf{P}(x, y)}{\partial y} = -\mathbf{P}(x, y)i\mathbf{H}(y). \tag{23}$$

(iii) Property 3:

$$\mathbf{P}(x, y) = \mathbf{P}(x, z)\mathbf{P}(z, y). \tag{24}$$

(iv) Property 4: in the case that $\mathbf{H}(x)$ and $\frac{\partial}{\partial x}\mathbf{H}(x)$ commute for every x such that $a < x < b$, then

$$\mathbf{P}(b, a) = \exp \int_a^b i\mathbf{H}(\xi)\, d\xi. \tag{25}$$

(v) Property 5 (the sum rule): let $\mathbf{P}_A(x, a) = \breve{\int}_a^x \exp\{i\mathbf{A}(\xi)\}\, d\xi$, then

$$\breve{\int}_a^x \exp\{i\mathbf{A}(\xi) + i\mathbf{B}(\xi)\}\, d\xi$$
$$= \mathbf{P}_A(x, a) \breve{\int}_a^x \exp\left\{\mathbf{P}_A^{-1}(\xi, a)i\mathbf{B}(\xi)\mathbf{P}_A(\xi, a)\right\} d\xi, \tag{26}$$

$$\breve{\int}_a^x \exp\{i\mathbf{A}(\xi) + i\mathbf{B}(\xi)\}\, d\xi$$
$$= \breve{\int}_a^x \exp\left\{\mathbf{P}_A(x, \xi)i\mathbf{B}(\xi)\mathbf{P}_A^{-1}(\xi, a)\right\} d\xi\, \mathbf{P}_A(x, a). \tag{27}$$

From the definition and the properties of product integrals, several different forms of the propagator can be defined. Each form emphasizes different aspects of propagation and allows a different interpretation.

2.3 The first form of the propagator

We represent a propagator for a general laterally heterogeneous medium from the definition of the product integral following Gilbert & Backus (1966):

$$\mathbf{P}(x, x_s) \equiv \lim_{L\to\infty} \prod_{l=1}^{L} \{\mathbf{I} + i\mathbf{H}(\xi_l)\zeta_l\} \qquad \text{(the first form)}, \tag{28}$$

where the range interval (x_s, x) is divided into L parts and $x_s \equiv x_0$ and $x \equiv x_L$. The coupling matrix $\mathbf{H}$ in eq. (28) consists of the matrices $\mathbf{K}$ and $\hat{\mathbf{B}}$. The off-diagonal matrix $\hat{\mathbf{B}}$ represents mode coupling between the different branches (i.e. intermode coupling) and the diagonal matrix $\mathbf{K}$

governs the phase variation of each mode. While the first form of the propagator allows only Born-approximation-type first-order intermode coupling in each subinterval (see fig. 3 of Marquering & Snieder 1995), it gives an exact solution with multiple coupling effects in total as the number of subintervals goes to infinity.

2.4 The second form of the propagator

According to our definition eq. (20), and assuming that the coefficient matrix **H** in a short subinterval is represented by the mean coefficient yields

$$\mathbf{H}(x) \simeq \mathbf{H}(\xi_l) \qquad \text{for} \qquad x_{l-1} \le x \le x_l. \tag{29}$$

Then, the $\prod$ approximant to the product integral (28) becomes (Gilbert & Backus 1966)

$$\mathbf{P}(x, x_s) \simeq \prod_{l=1}^{L} \exp\{i\mathbf{H}(\xi_l)\zeta_l\} \qquad \text{(the second form).} \tag{30}$$

The propagator **P** in eq. (30) denotes all possible higher order coupling along the same dispersion branch and between the different branches in each subinterval. Also note that the propagator **P** governs both the amplitude and the phase of each mode. In the limit that $L \to \infty$, the first and second are identical as they differ only by unimportant higher order terms. This is obvious when $\exp[i\mathbf{H}(\xi_l)\zeta_l]$ is replaced by its series expansion.

2.5 The third form of the propagator

We can also express the coupled-mode propagator in another form by using the relation between the unitary operator and the Hermitian operator and the relation

$$\tan\frac{\theta}{2} = \frac{1}{i}\frac{e^{i\theta} - 1}{e^{i\theta} + 1}$$

(Morse & Feshbach 1953, Vol. I pp. 84–85):

$$\mathbf{P}(x_r, x_s) = \frac{\mathbf{I} + i\mathbf{Z}}{\mathbf{I} - i\mathbf{Z}} \qquad \text{(the third form),} \tag{31}$$

where the operator

$$\mathbf{Z}(x_r, x_s) = \tan\left\{\frac{1}{2}\sum_{l=1}^{L}(x_l - x_{l-1})\mathbf{H}(\xi_l)\right\} \tag{32}$$

and **I** is the identity operator. If we consider the $\hat{\mathbf{c}}(x_s)$ as an input and $\hat{\mathbf{c}}(x_r)$ as an output, the operator **Z** can be interpreted as an impedance as a result of lateral heterogeneities located between x_s and x_r. The propagator $\mathbf{P}(x_r, x_s)$ itself has the form of a scattering operator and includes both forward and backscattered modes. This form of the propagator permits analogies to be drawn between a lumped parameter equivalent circuit for a transmission line and propagation in a heterogeneous waveguide. Schwinger & Saxon (1968), employing a slightly different notation, have a good discussion of the transmission line analogy.

2.6 The fourth form of the propagator

From the definition eq. (20) and property 5, the sum rule eq. (26), the propagator is represented as

$$\begin{aligned}\mathbf{P}(x, x_s) &= \breve{\int}_{x_s}^{x} \exp\{i\mathbf{H}(\xi)\}\, d\xi \\ &= \breve{\int}_{x_s}^{x} \exp\{i\mathbf{K}(\xi) + i\hat{\mathbf{B}}(\xi)\}\, d\xi \\ &= \mathbf{P}_K(x, x_s)\breve{\int}_{x_s}^{x} \exp\left\{\mathbf{P}_K^{-1}(\xi, x_s) i\hat{\mathbf{B}}(\xi)\mathbf{P}_K(\xi, x_s)\right\} d\xi,\end{aligned} \tag{33}$$

where we have substituted **K** for **A** and $\hat{\mathbf{B}}$ for **B** in eq. (26).

Because the diagonal matrices $\mathbf{K}(x)$ and $\frac{\partial}{\partial x}\mathbf{K}(x)$ are commutative for every x, $\mathbf{P}_K$ can be rewritten by property 4 eq. (25):

$$\mathbf{P}_K(x, x_s) = \exp\left\{i\int_{x_s}^{x} \mathbf{K}(\xi)\, d\xi\right\}. \tag{34}$$

The propagator of eq. (33) becomes

$$\mathbf{P}(x, x_s) = \exp\left\{i\int_{x_s}^{x} \mathbf{K}(\xi)\, d\xi\right\} \breve{\int}_{x_s}^{x} \exp\{\mathbf{B}(\xi)\}\, d\xi, \tag{35}$$

where the following relation is used:

$$\begin{aligned}\left\{\mathbf{P}_K^{-1}(x,x_s)i\hat{\mathbf{B}}(x)\mathbf{P}_K(x,x_s)\right\}_{\alpha\beta}\\
&= \sum_{\gamma,\nu}\exp\left\{-i\int_{x_s}^{x}k_\alpha(\xi)\,d\xi\right\}\delta_{\alpha\gamma}i\hat{B}_{\gamma\nu}\exp\left\{i\int_{x_s}^{x}k_\nu(\xi)\,d\xi\right\}\delta_{\nu\beta}\\
&= i\hat{B}_{\alpha\beta}\exp\left\{i\int_{x_s}^{x}\left(k_\beta-k_\alpha(\xi)\right)d\xi\right\}\\
&= B_{\alpha\beta}.\end{aligned} \tag{36}$$

By the definition of the product integral, the second term on the right hand side of eq. (35) is written as

$$\check{\int}_{x_s}^{x}\exp\{\mathbf{B}(\xi)\}\,d\xi \approx \prod_{l=1}^{L}\exp\{\mathbf{B}(\xi_l)\zeta_l\}. \tag{37}$$

Finally, the fourth form of the propagator becomes

$$\begin{aligned}\mathbf{P}(x,x_s) &= \mathbf{U}^K(x_L,x_{L-1})\mathbf{U}^B(x_L,x_{L-1})\cdots\\
&\times \mathbf{U}^K(x_2,x_1)\mathbf{U}^B(x_2,x_1)\mathbf{U}^K(x_1,x_s)\mathbf{U}^B(x_1,x_s)\\
&\qquad \text{(the fourth form)},\end{aligned} \tag{38}$$

where the operators $\mathbf{U}^K$ and $\mathbf{U}^B$ for the subinterval $[x_{l-1}, x_l]$ are defined as

$$\mathbf{U}^K(x_l,x_{l-1}) = \exp\left\{i\int_{x_{l-1}}^{x_l}\mathbf{K}(\xi)\,d\xi\right\}, \tag{39}$$

$$\mathbf{U}^B(x_l,x_{l-1}) = \exp\{\mathbf{B}(\xi_l)\zeta_l\}. \tag{40}$$

The operator $\mathbf{U}^K(x_l, x_{l-1})$ represents the phase integral of each wavenumber in the subinterval and the operator $\mathbf{U}^B(x_l, x_{l-1})$ denotes the transition from one mode to another mode. It is computed from the lateral heterogeneities in the subinterval and applied at the end of each subinterval. The magnitude $U^B_{\alpha_1\alpha_2}(x_l, x_{l-1})$ is an amplitude for the transition from a mode α_1 to α_2, i.e. the transition amplitude. To interpret the product integral of the propagator, we present a schematic figure showing two possible paths with fixed starting and ending points in wavenumber space in Fig. 1. The horizontal thick solid lines denote the phase integral and the vertical arrows represent the spectral transition.

The product integral representation for the propagator eq. (38) is very important, because it provides a clear physical picture of the mode coupling process when it is carefully interpreted. Mode coupling in a laterally heterogeneous medium is a continuous process. However, eq. (38) approximates the continuous process in the heterogeneous medium as a sequence of processes in homogeneous subintervals within which the infinitesimal evolution operators $\mathbf{U}^{K,B}$ are defined. The evolution operator $\mathbf{U}^B$ allows for transitions (coupling) between modes with different local eigenwavenumbers. The operator $\mathbf{U}^K$ is a diagonal operator of phase integrals of the form $\int_{x_{l-1}}^{x_l} ik_\alpha(\xi)\,d\xi$. Notice that the phase integral $\int_{x_s}^{x_r} ik(\xi)\,d\xi$ is a functional of the discrete wavenumber functions $k_\alpha(\xi)$, i.e. the value of the phase integral does not depend on the intermediate variable ξ ($x_s < \xi < x_r$). Rather it depends on the shape of the wavenumber function $k(\xi)$ and the endpoints (x_s, x_r). Because of

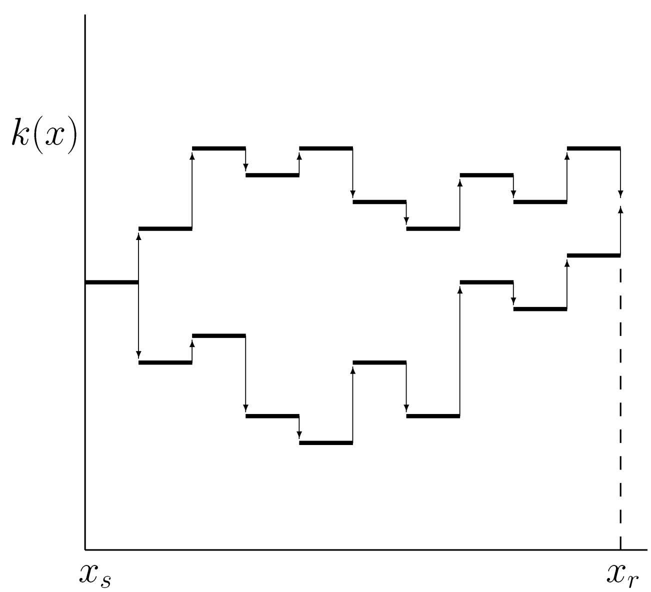


Figure 1. Two possible paths are shown with fixed starting and ending points in the discrete wavenumber space. The horizontal thick solid lines denote the phase integral $\mathbf{U}^K(x_l, x_{l-1})$ and the vertical arrows represent the spectral transition $\mathbf{U}^B(x_l, x_{l-1})$.

the presence of mode coupling (transitions) through the presence of the operator $\mathbf{U}^B$, the propagator $\mathbf{P}$ represents the phase integral computed along all possible paths in the discrete functional space of the wavenumber $\int_{x_s}^{x_r} ik(\xi)\,d\xi$.

For a slowly varying structure, the coupling matrix $\hat{\mathbf{B}}$ may be disregarded, then the coupled-mode propagator becomes

$$\begin{aligned}\mathbf{P}(x, x_s) &= \lim_{L\to\infty}\prod_{l=1}^{L}\exp\left\{\int_{x_{l-1}}^{x_l} i\mathbf{K}(\xi)\,d\xi\right\}\\ &= \exp\int_{x_s}^{x}\{i\mathbf{K}(\xi)\,d\xi\}.\end{aligned} \tag{41}$$

The coupled-mode propagator of eq. (41) denotes the phase integral of each wavenumber from the source point to the receiver point without mode–mode transitions, i.e. the JWKB approximation for surface waves (Woodhouse 1974).

2.7 The solutions for the fields in terms of the propagators

In terms of the coupled-mode propagator eq. (30), the solution of the evolution eq. (5) is represented as

$$\hat{\mathbf{c}}^{(0)}(x) = \mathbf{P}(x, x_s)\hat{\mathbf{c}}^{(0)}(x_s). \tag{42}$$

The second term on the right hand side of eq. (6) represents the excitation of the scattered field from scattering of the coherent field, i.e. eq. (6) is an inhomogeneous equation with the second term as the source. The propagator can also be used to solve the system with a source (Gantmacher 1959):

$$\hat{\mathbf{d}}(x) = \mathbf{P}(x, x_0)\hat{\mathbf{d}}(x_0) + \int_{x_0}^{x}\mathbf{P}(x, \xi)\mathbf{g}(\xi)\,d\xi, \tag{43}$$

where the source term is

$$\begin{aligned}\mathbf{g}(x) &= \mathbf{S}(x)\hat{\mathbf{c}}(x)\\ &= (\hat{\mathbf{D}} + i\hat{\mathbf{E}}\hat{\mathbf{B}})\hat{\mathbf{c}}(x).\end{aligned} \tag{44}$$

The source term $\mathbf{g}(x)$ in eq. (43) can be interpreted as the stochastic effective source of the scattered field, which represents the excitation of the scattered field as a result of the coherent field scattered from the stochastic roughness located at x. The transformed evolution eq. (43) for the $\mathcal{O}(\varepsilon)$ system was used to derive the integral equation for the propagator in the $\mathcal{O}(\varepsilon)$ stochastic medium in the next section.

3 LIPPMANN–SCHWINGER, DYSON, AND BETHE–SALPETER EQUATIONS

Park & Odom (1999) showed that the propagator for the stochastic system satisfies an integral equation,

$$\boxed{\tilde{\mathbf{P}}(x, x_0) = \mathbf{P}(x, x_0) - \int_{x_0}^{x}\mathbf{P}(x, \xi)\mathbf{S}(\xi)\tilde{\mathbf{P}}(\xi, x_0)\,d\xi}\,, \tag{45}$$

where $\tilde{\mathbf{P}}$ is the coupled-mode propagator for the field in the medium with random rough interfaces ($\gamma(x) \neq 0$) and $\mathbf{P}$ is the coupled-mode propagator for the range-dependent medium in the absence of the stochastic roughness ($\gamma(x) = 0$). The operator $\mathbf{S}$, defined in eq. (44) represents rough surface scattering at ξ, i.e. it converts the coherent field $\hat{\mathbf{c}}$ to the scattered field $\hat{\mathbf{d}}$. Note that eq. (45) includes a scattering operator $\mathbf{S}(x)$ instead of the scalar function for random fluctuations commonly included in the Lippmann–Schwinger integral equation for scalar wave (or potential) scattering.

By applying formal iteration Park & Odom (1999) derived a solution to eq. (45):

$$\begin{aligned}\tilde{\mathbf{P}}(x, x_0) = {}& \mathbf{P}(x, x_0) - \int_{x_0}^{x}\mathbf{P}(x, \xi_1)\mathbf{S}(\xi_1)\mathbf{P}(\xi_1, x_0)\,d\xi_1\\ &+ (-1)^2\int_{x_0}^{x}\mathbf{P}(x, \xi_2)\mathbf{S}(\xi_2)\mathbf{P}(\xi_2, \xi_1)\mathbf{S}(\xi_1)\mathbf{P}(\xi_1, x_0)\,d\xi_1\,d\xi_2\\ &+ (-1)^3\int_{x_0}^{x}\mathbf{P}(x, \xi_3)\mathbf{S}(\xi_3)\mathbf{P}(\xi_3, \xi_2)\mathbf{S}(\xi_2)\mathbf{P}(\xi_2, \xi_1)\\ &\quad\times\mathbf{S}(\xi_1)\mathbf{P}(\xi_1, x_0)d\xi_1\,d\xi_2\,d\xi_3\\ &\ldots.\end{aligned} \tag{46}$$

Note that retaining the first two terms on the right hand side of eq. (46) is the Born approximation for $\mathbf{P}(x,\ x_0)$ (the single-scattering approximation). The averaged quantity of the Born approximation vanishes for zero-mean random fluctuations. On the other hand, averaging the formal perturbation series eq. (46) yields the non-vanishing mean propagator $\langle\tilde{\mathbf{P}}\rangle_E$, which can be related to the mean coherent field (Park & Odom 1999).

Frisch (1968) derived the Dyson equation for the ensemble averaged Green's function for scalar wave scattering by diagram methods. However, the formal series solution of the propagator, eq. (46) includes the scattering operator $\mathbf{S}(x)$ instead of a scalar random function. To derive the mean propagator, we need to express the explicit correlation relations between components of the scattering operator at different points. Because the scattering operator $\mathbf{S}(x)$ is linearly dependent on the random roughness function $\gamma(x)$, we can decompose the random operator $\mathbf{S}(x)$ into a product of the zero-mean stochastic process $\gamma(x)$ and a deterministic operator $\hat{\mathbf{S}}$ depending on the reference propagator $\mathbf{P}$. The averaged quantities of the zero-mean random operator $\mathbf{S}$, such as the l^{th} moment $\langle \mathbf{S}(x_1)\mathbf{S}(x_2)\cdots\mathbf{S}(x_l)\rangle_E$, also depend on the configuration of the points $x_1, x_2, \ldots, x_l$ in the same way as the corresponding averaged quantities of the random function $\gamma(x)$. Therefore, by means of the cluster expansions of the stochastic process $\gamma(x)$ given by Γ eq. (A3), we can also expand the averaged quantities of the random operator $\mathbf{S}(x)$:

$$\begin{aligned}
\langle \mathbf{S}(x_1)\mathbf{S}(x_2)\rangle_E &= \Gamma_2(x_1,x_2)\hat{\mathbf{S}}(x_1)\hat{\mathbf{S}}(x_2)\\
\langle \mathbf{S}(x_1)\mathbf{S}(x_2)\mathbf{S}(x_3)\rangle_E &= \Gamma_3(x_1,x_2,x_3)\hat{\mathbf{S}}(x_1)\hat{\mathbf{S}}(x_2)\hat{\mathbf{S}}(x_3)\\
\langle \mathbf{S}(x_1)\mathbf{S}(x_2)\mathbf{S}(x_3)\mathbf{S}(x_4)\rangle_E &= \{\Gamma_2(x_1,x_2)\Gamma_2(x_3,x_4)+\Gamma_2(x_1,x_3)\Gamma_2(x_2,x_4)\\
&\quad+\ \Gamma_2(x_1,x_4)\Gamma_2(x_2,x_3)+\Gamma_4(x_1,x_2,x_3,x_4)\}\\
&\quad\times\hat{\mathbf{S}}(x_1)\hat{\mathbf{S}}(x_2)\hat{\mathbf{S}}(x_3)\hat{\mathbf{S}}(x_4)\\
&\ \ \vdots
\end{aligned}\tag{47}$$

A summation over each interface should be performed for the cluster expansion eq. (47) when the model includes multiple rough interfaces. Note that, unlike the random scalar function, the scattering operator $\hat{\mathbf{S}}$ and the propagator $\mathbf{P}$ must be ordered for the averaged quantity. Averaging the formal series eq. (46) yields the expression for the mean propagator $\langle\tilde{\mathbf{P}}\rangle_E$ in terms of the cluster expansions of the random operator $\mathbf{S}(x)$:

$$\begin{aligned}
\langle\tilde{\mathbf{P}}(x,x_0)\rangle_E &= \mathbf{P}(x,x_0)\\
&\quad+(-1)^2\int_{x_0}^{x}\mathbf{P}(x,\xi_2)\hat{\mathbf{S}}(\xi_2)\mathbf{P}(\xi_2,\xi_1)\hat{\mathbf{S}}(\xi_1)\mathbf{P}(\xi_1,x_0)\Gamma_2(\xi_1,\xi_2)\,d\xi_1\,d\xi_2\\
&\quad+(-1)^3\int_{x_0}^{x}\mathbf{P}(x,\xi_3)\hat{\mathbf{S}}(\xi_3)\mathbf{P}(\xi_3,\xi_2)\hat{\mathbf{S}}(\xi_2)\mathbf{P}(\xi_2,\xi_1)\\
&\qquad\times\hat{\mathbf{S}}(\xi_1)\mathbf{P}(\xi_1,x_0)\Gamma_3(\xi_1,\xi_2,\xi_3)\,d\xi_1\,d\xi_2\,d\xi_3\\
&\quad+(-1)^4\int_{x_0}^{x}\mathbf{P}(x,\xi_4)\hat{\mathbf{S}}(\xi_4)\mathbf{P}(\xi_4,\xi_3)\hat{\mathbf{S}}(\xi_3)\mathbf{P}(\xi_3,\xi_2)\\
&\qquad\times\hat{\mathbf{S}}(\xi_2)\mathbf{P}(\xi_2,\xi_1)\hat{\mathbf{S}}(\xi_1)\mathbf{P}(\xi_1,x_0)\{\Gamma_2(\xi_1,\xi_2)\Gamma_2(\xi_3,\xi_4)\\
&\qquad+\Gamma_2(\xi_1,\xi_3)\Gamma_2(\xi_2,\xi_4)+\Gamma_2(\xi_1,\xi_4)\Gamma_2(\xi_2,\xi_3)\Gamma_4(\xi_1,\xi_2,\xi_3,\xi_4)\}\\
&\qquad d\xi_1\,d\xi_2\,d\xi_3\,d\xi_4\\
&\quad\ldots
\end{aligned}\tag{48}$$

Note that averaging the second term on the right hand side of eq. (46) vanishes for the zero-mean random operator, i.e. $\langle\mathbf{S}(x)\rangle_E = 0$. For the zero-mean Gaussian stochastic process $\gamma(x)$, all statistical moments of odd order vanish and statistical moments of order $2n$ can be written as sums of $\frac{(2n)!}{2^n n!}$ terms, each of which is a product of two-point correlation functions. Then the mean propagator becomes

$$\begin{aligned}
\langle\tilde{\mathbf{P}}(x,x_0)\rangle_E &= \mathbf{P}(x,x_0)\\
&\quad+\int_{x_0}^{x}\mathbf{P}(x,\xi_2)\hat{\mathbf{S}}(\xi_2)\mathbf{P}(\xi_2,\xi_1)\hat{\mathbf{S}}(\xi_1)\mathbf{P}(\xi_1,x_0)\Gamma_2(\xi_1,\xi_2)\,d\xi_1\,d\xi_2\\
&\quad+\int_{x_0}^{x}\mathbf{P}(x,\xi_4)\hat{\mathbf{S}}(\xi_4)\mathbf{P}(\xi_4,\xi_3)\hat{\mathbf{S}}(\xi_3)\mathbf{P}(\xi_3,\xi_2)\\
&\qquad\times\hat{\mathbf{S}}(\xi_2)\mathbf{P}(\xi_2,\xi_1)\hat{\mathbf{S}}(\xi_1)\mathbf{P}(\xi_1,x_0)\\
&\qquad\times\{\Gamma_2(\xi_1,\xi_2)\Gamma_2(\xi_3,\xi_4)+\Gamma_2(\xi_1,\xi_3)\Gamma_2(\xi_2,\xi_4)+\Gamma_2(\xi_1,\xi_4)\Gamma_2(\xi_2,\xi_3)\}\\
&\qquad d\xi_1\,d\xi_2\,d\xi_3\,d\xi_4\\
&\quad\ldots
\end{aligned}\tag{49}$$

3.1 The diagram method and the Dyson equation for the mean propagator

To give a graphic idea of the structure of the expansions eqs (48) and (49), we represent their elements by Feynman diagrams. The diagram method was introduced by Feynman to quantum electrodynamics to provide a simple way for handling all-order formal perturbation series of the Green's function. The method owes its success to its compact form as compared with the analytical representation. We will apply the diagram methods to our formal perturbation series for the propagator, which is defined for the vector representations of the wavefields in terms of local modes.

We point out that the following diagram representation is a calculation tool and not just a graphical representation of the equations. Diagrams can be manipulated by well-defined rules based on their topology. The diagram method allows us to represent the terms in equations for the mean and covariance of the propagator in the form of an integral equation with a kernel, which would be extremely difficult to do using ordinary analytical methods. Their application to classical problems is far from unknown. Tatarskii (1971) made extensive use of diagram methods in his monograph on sound and electromagnetic wave propagation in a turbulent atmosphere.

The diagram method can be introduced in a very elementary way. Following Frisch (1968), Bass & Fuks (1979) and Rytov *et al.* (1989), the diagram method for the propagator $\tilde{\mathbf{P}}$ is developed. First, the following connection is introduced for the representation of bare diagrams (Frisch 1968; Rytov *et al.* 1989).

(i) The propagator for the deterministic medium (the deterministic propagator) is represented by a thin solid line:

$$\mathbf{P}(x, x_0) = \quad x \overline{\qquad\qquad\qquad} x_0 \quad . \tag{50}$$

(ii) The random operator $(-L_1)$ is represented by a dot:

$$-L_1 = \quad \bullet \quad . \tag{51}$$

Then, the perturbation series solution of the propagator in eq. (46) is represented as

$\tilde{\mathbf{P}}(x, x_0) =$

(52)

The series of eq. (52) has a multiple-scattering physical interpretation. The nth term corresponds to an nth order (n times) scattered wave, which propagates freely from x_0 to ξ_1, is scattered at ξ_1 by roughness, propagates freely to ξ_2, is scattered at ξ_2 and so on. Here, the free propagation must be interpreted as propagation without scattering.

We now turn back to the calculation of the mean propagator $\langle\tilde{\mathbf{P}}\rangle_E$. The cluster expansions of the mean propagator can be represented by a dressed diagram with the following conventions (Frisch 1968).

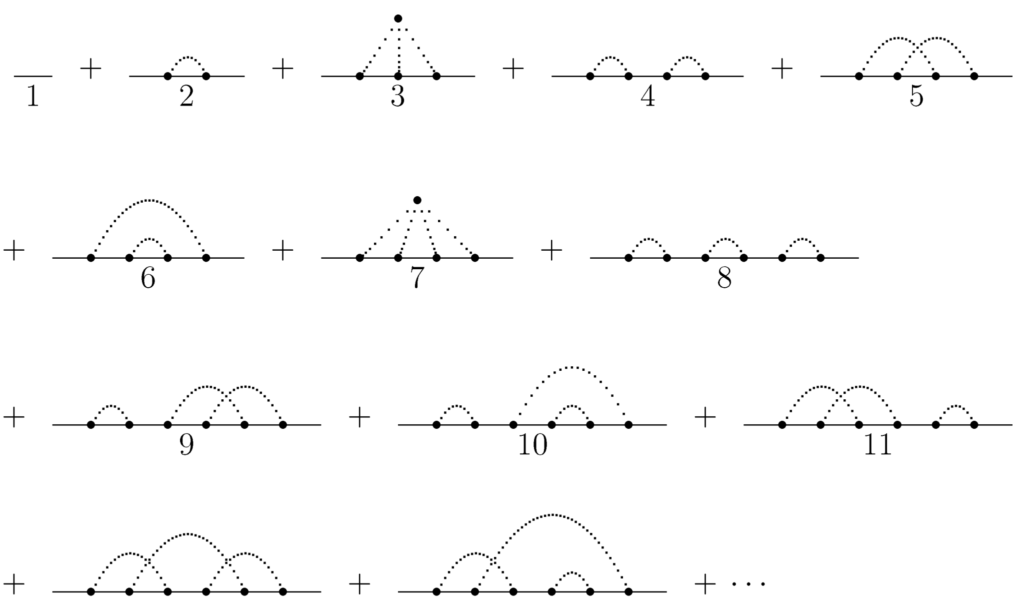
(i) Points belonging to a given cluster are connected by dotted lines.

(ii) To every bare diagram involving l factors **S** is associated as many dressed diagrams as there are different partitions of the set $x_1, x_2, \ldots, x_l$ into clusters of at least two points.

(iii) To calculate a dressed diagram, the solid lines are replaced by deterministic propagators, the cluster of dotted lines ending at $\xi_1, \ldots, \xi_l$ by factors $\hat{\mathbf{S}}(\xi_1)\cdots\hat{\mathbf{S}}(\xi_l)\Gamma_l(\xi_1, \ldots, \xi_l)$ and integration is performed over all intermediate points.

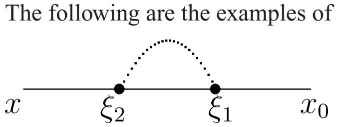
The expansion of the mean propagator eq. (48) can now be written in terms of dressed diagrams:

$\left\langle \tilde{\mathbf{P}} \right\rangle_E =$

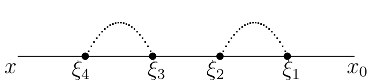


(53)

The following are the examples of some simple dressed diagram representations used in eq. (35):

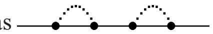


$$= \int \mathbf{P}(x, \xi_2)\hat{\mathbf{S}}(\xi_2)\mathbf{P}(\xi_2, \xi_1)\hat{\mathbf{S}}(\xi_1)\mathbf{P}(\xi_1, x_0)\Gamma_2(\xi_1, \xi_2)\, d\xi_1\, d\xi_2, \tag{54}$$



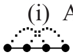
$$= \int \mathbf{P}(x, \xi_4)\hat{\mathbf{S}}(\xi_4)\mathbf{P}(\xi_4, \xi_3)\hat{\mathbf{S}}(\xi_3)\mathbf{P}(\xi_3, \xi_2)\hat{\mathbf{S}}(\xi_2)\mathbf{P}(\xi_2, \xi_1)\hat{\mathbf{S}}(\xi_1)\mathbf{P}(\xi_1, x_0)$$
$$\times\, \Gamma_2(\xi_3, \xi_4)\Gamma_2(\xi_1, \xi_2)\, d\xi_1\, d\xi_2\, d\xi_3\, d\xi_4. \tag{55}$$

For a zero-mean Gaussian roughness $\gamma(x)$, there are no clusters of more than two points, hence diagrams 3 and 7 in eq. (53) disappear. Note that the number of terms of nth order series in eq. (53) increases very rapidly with n, at least as $\frac{(2n)!}{2^n n!}$. This is another reason for the divergence of the perturbation expansion of the mean propagator (Frisch 1968). A technique for making such expansions more uniformly valid will be briefly discussed at the end of this section.

When considering the analytical expression in eq. (55) for a diagram such as , it can be written as the product of the following five diagrams:

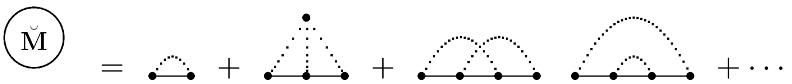
(56)

It is obvious that this factorization property is related to the topological structure of the diagram. To obtain an integral equation for $\tilde{\mathbf{P}}$ with a kernel consisting of an infinite series, we introduce the following definitions based on the topology of the diagrams.

(i) A diagram without terminals is a diagram that has been stripped of its external solid lines, such as  of diagram 2 in eq. (53) or  of diagram 5 in eq. (53).

(ii) A diagram without terminals is connected if it cannot be cut into two or more diagrams, without cutting any dotted lines, in other words if it is not factorizable. Diagrams 2, 3, 5, 6, 7, 12 and 13, after being stripped of their external solid lines, are connected, while diagrams 4, 8, 9, 10 and 11 are not connected. Therefore, any unconnected diagram can be factored into lower order connected diagrams.

(iii) The mass operator (the name was borrowed from quantum field theory) is defined as the sum of all possible connected diagrams contributing to $\langle\tilde{\mathbf{P}}\rangle_E$. It is denoted by $\check{\mathbf{M}}$ or the symbol ($\check{\mathbf{M}}$):



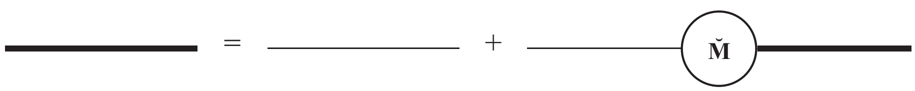
(57)

(iv) The symbol for the mean propagator $\langle\tilde{\mathbf{P}}\rangle_E$ is introduced as

$\langle\tilde{\mathbf{P}}(x, x_0)\rangle_E =$ 

. (58)

When all unconnected diagrams are factored into connected diagrams, the diagrammatic expansion of $\langle\tilde{\mathbf{P}}\rangle_E$ in eq. (53) may be represented by the sum of all possible connected diagrams. A graphical representation of the relation between the mean propagator and the mass operator is



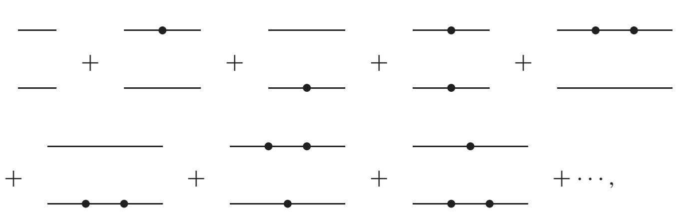
(59)

Eq. (59) is an integral equation for $\langle\tilde{\mathbf{P}}\rangle_E$ with the kernel of the mass operator, which is called the Dyson equation. The analytical form of the Dyson equation is

$$\boxed{\langle\tilde{\mathbf{P}}(x, x_0)\rangle_E = \mathbf{P}(x, x_0) + \int_{x_0}^{x} \mathbf{P}(x, \xi_1)\check{\mathbf{M}}(\xi_1, \xi_2)\langle\tilde{\mathbf{P}}(\xi_2, x_0)\rangle_E \, d\xi_1 \, d\xi_2}\,. \tag{60}$$

3.2 The Bethe–Salpeter equation for the covariance of the propagator

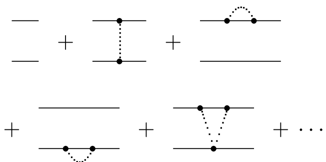
We investigate the second statistical moment of the propagator $\langle\tilde{\mathbf{P}}^\dagger\tilde{\mathbf{P}}\rangle_E$, i.e. the covariance of two propagators (fields). The importance of the propagator covariance is that it enables us to synthesize the envelope of the mean squared signals (e.g. envelopes of bottom interacting seismo-acoustic signals, the seismic coda envelope, etc.) and analyse the effect of the multiple rough surface scattering on the envelope decay. First, the product of the propagator $\tilde{\mathbf{P}}$ and its Hermitian conjugate is written in terms of the bare double diagrams (Frisch 1968):

$\tilde{\mathbf{P}}^\dagger(x, x_0)\tilde{\mathbf{P}}(x', x_0') =$ 

(61)

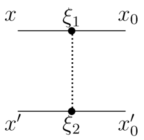
where we make the convention that each double diagram is the product of the propagator corresponding to the lower line and of the Hermitian conjugate of the propagator corresponding to the upper line. For the propagator covariance, which is the mean double propagator, a similar expansion holds in terms of dressed double diagrams:

$$\left\langle \tilde{\mathbf{P}}^{\dagger}(x, x_0)\tilde{\mathbf{P}}(x', x_0')\right\rangle_E =$$

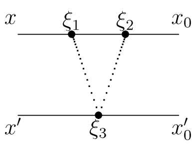


(62)

We also give examples of the analytical representation for some simple dressed double diagrams used in eq. (62):



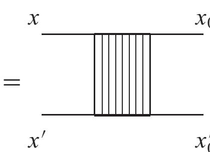
$$= \int \mathbf{P}^{\dagger}(x, \xi_1)\hat{\mathbf{S}}^{\dagger}(\xi_1)\mathbf{P}^{\dagger}(\xi_1, x_0) \times \mathbf{P}(x', \xi_2)\hat{\mathbf{S}}(\xi_2)\mathbf{P}(\xi_2, x_0') \times \Gamma_2(\xi_1, \xi_2)\, d\xi_1\, d\xi_2 \tag{63}$$



$$= \int \mathbf{P}^{\dagger}(x, \xi_1)\hat{\mathbf{S}}^{\dagger}(\xi_1)\mathbf{P}^{\dagger}(\xi_1, \xi_2)\hat{\mathbf{S}}^{\dagger}(\xi_2)\mathbf{P}^{\dagger}(\xi_2, x_0)\mathbf{P}(x', \xi_3)\hat{\mathbf{S}}(\xi_3)\mathbf{P}(\xi_3, x_0')\Gamma_2(\xi_1, \xi_3)\Gamma_2(\xi_2, \xi_3)\, d\xi_1\, d\xi_2\, d\xi_3. \tag{64}$$

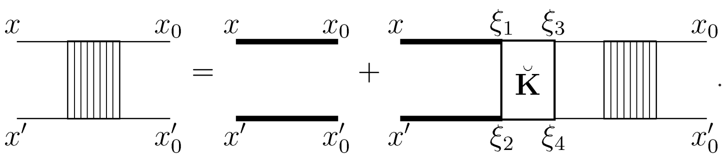
Similarly, all unconnected double diagrams are represented by the connected double diagram by the factorization property and all double diagrams that are not connected between the upper line and the lower line can be represented by the product $\langle \tilde{\mathbf{P}}^{\dagger}\rangle_E \langle \tilde{\mathbf{P}}\rangle_E$. We can remove the diagrams that are not connected between the upper line and the lower line, and focus just on the covariance. By introducing the symbol for the propagator covariance,

$$\langle \tilde{\mathbf{P}}^{\dagger}(x, x_0)\tilde{\mathbf{P}}(x', x_0')\rangle_E =$$



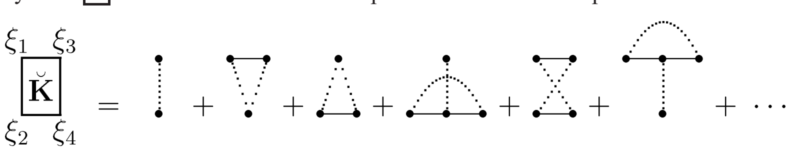
, (65)

the diagrammatic expansion of the propagator covariance is written as an integral equation with a kernel that is the sum of all possible connected double diagrams contributing to $\langle \tilde{\mathbf{P}}^{\dagger}\tilde{\mathbf{P}}\rangle_E$:



. (66)

Eq. (66) is called the Bethe–Salpeter equation and its kernel is the intensity operator (Frisch 1968; Rytov *et al.* 1989) denoted by $\breve{\mathbf{K}}$ or the symbol $\boxed{\breve{\mathbf{K}}}$. The infinite series of $\breve{\mathbf{K}}$ represents the sum of all possible connected double diagrams. Its diagram representation is



(67)

The analytical expression for the Bethe–Salpeter equation is

$$\left\langle \tilde{\mathbf{P}}^{\dagger}(x, x_0)\tilde{\mathbf{P}}(x', x_0')\right\rangle_E = \langle \mathbf{P}^{\dagger}(x, x_0)\rangle_E \langle \mathbf{P}(x', x_0')\rangle_E + \int_{x_0}^{x} \langle \mathbf{P}^{\dagger}(x, \xi_1)\rangle_E \langle \mathbf{P}(x, \xi_2)\rangle_E \breve{\mathbf{K}}(\xi_1, \xi_3; \xi_2, \xi_4)\left\langle \tilde{\mathbf{P}}^{\dagger}(\xi_3, x_0)\tilde{\mathbf{P}}(\xi_4, x_0')\right\rangle_E d\xi_1\, d\xi_2\, d\xi_3\, d\xi_4. \tag{68}$$

Eq. (68) corrects a typographical error in eq. (87) of Park & Odom (1999). As an aside, we note that our symbol for the propagator covariance eq. (65) intentionally differs from Tatarskii's (1971) symbol for the same quantity. This is because Tatarskii assumed Gaussian random perturbations, with the result that all odd-order cluster expansions are zero. We have not made the assumption of Gaussian perturbations. Consequently we have retained odd-order cluster expansions and the accompanying diagrams. The symbol eq. (65) also does not imply the ladder approximation, an approximation we have not discussed in this work.

The Dyson equation and the Bethe–Salpeter equation enable us to analyse the phenomenon of elastic wave multiple scattering. However, the expansions obtained in this section were shown to be divergent by Frisch (1968). Moreover, he also showed that these expansions contain secular terms, which restrict the validity of these asymptotic expansions to small arguments. Because $\langle \mathbf{S}(x_1)\mathbf{S}(x_2)\cdots\mathbf{S}(x_l)\rangle_E$ is the sum of

$\frac{(2l)!}{2^l l!}$ $(= (2l-1)!!)$ two-point correlation functions, this number increases rapidly with increasing l, and this is another reason why eqs (53) and (62) are divergent. Finite approximations for the mass operator $\check{\mathbf{M}}$ and the intensity operator $\check{\mathbf{K}}$ correspond to partial summations of the complete perturbation series, which retain terms of any order, and may therefore constitute a non-secular uniform approximation (Frisch 1968). A technique for making such expansions more uniformly valid, is renormalization. An example of the renormalization technique applied to the Dyson equation is the first-order smoothing approximation (or Bourret approximation, Bourret 1961). The application of renormalization techniques is beyond the scope of this paper, but further work on the renormalization will be required to solve these equations and analyse multiple-scattering effects for specific examples.

4 CONCLUSIONS

We have reviewed the propagator and product integral for elastic wave propagation and derived three additional representations of the propagator beyond the original form of Gilbert & Backus (1966). These additional forms include, a lateral impedance representation and a representation that naturally reduces to the JWKB approximation for slowly varying structure with no cross-branch coupling.

Diagram techniques are used to efficiently derive the Dyson and Bethe–Salpeter equations for the propagator in a random medium. These two equations govern the mean and covariance of the propagator, respectively. As they stand the Dyson and Bethe–Salpeter equations probably need renormalization to be applied to strongly heterogeneous media, but may be adequate for weakly random media.

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APPENDIX A: DEFINITIONS AND NOTATION FOR ROUGH LAYER BOUNDARY CHARACTERIZATION

In this appendix, we provide a brief description of our model as well as some definitions and notation for the quantities used in the main text. For more detail we refer the reader to Park & Odom (1999).

Our model is assumed to be a laterally heterogeneous layered half-space, and may contain both fluid and solid layers. We assume that the model can be characterized by some n-layered reference structure with deterministic but laterally varying thickness layers with boundaries

$h_n^0(x)$. To represent the stochastic part of the medium, we define the boundary roughness function $\gamma_n(x)$ as the deviation of the n^{th} boundary function of the exact structure $h_n(x)$ from the boundary function of the reference structure $h_n^0(x)$:

$$h_n(x) \equiv h_n^0(x) + \varepsilon\gamma_n(x), \tag{A1}$$

where ε ($\ll 1$) is a relative amplitude of the rms height of roughness compared to the overall variation of the reference boundary. From now on, we drop the subscript n. The function $h^0(x)$ is deterministic and assumed to be known. We assume the roughness of the layer boundaries can be characterized by some correlation length scale.

The statistical moments and parameters of the random roughness function $\gamma(x)$ relate directly to those of the boundary function $h(x)$ because the reference boundary function $h^0(x)$ is deterministic and assumed known.

We summarize the statistical properties of the stochastic process $\gamma(x)$. Most of the properties are quite standard, but we make use of them in Section 3.

(i) $\gamma(x)$ is assumed a zero-mean process:

$$\langle\gamma\rangle_E = 0, \tag{A2}$$

where $\langle ... \rangle_E$ indicates ensemble average over all possible realizations.

(ii) The averaged quantities $\langle\gamma(x_1)\gamma(x_2)\gamma(x_l)\rangle_E$ depend on the configuration of the points $x_1, x_2, \ldots, x_l$ because for random media with a correlation length (a scale length of heterogeneity) a, the values of γ at points of separation larger than the correlation length a are uncorrelated. To express the dependence of the averaged quantity on the correlation length, we introduce the l-point correlation functions $\Gamma_l(x_1, x_2, \ldots, x_l)$ and we expand these averaged quantities in the following cluster expansions in terms of the l-point correlation functions (Frisch 1968; Nayfeh 1973):

$$\begin{aligned}
\langle\gamma(x_1)\gamma(x_2)\rangle_E &= \Gamma_2(x_1, x_2)\\
\langle\gamma(x_1)\gamma(x_2)\gamma(x_3)\rangle_E &= \Gamma_3(x_1, x_2, x_3)\\
\langle\gamma(x_1)\gamma(x_2)\gamma(x_3)\gamma(x_4)\rangle_E &= \Gamma_2(x_1, x_2)\Gamma_2(x_3, x_4) + \Gamma_2(x_1, x_3)\Gamma_2(x_2, x_4)\\
&\quad + \Gamma_2(x_1, x_4)\Gamma_2(x_2, x_3) + \Gamma_4(x_1, x_2, x_3, x_4)\\
&\;\;\vdots\\
\langle\gamma(x_1)\gamma(x_2)\cdots\gamma(x_l)\rangle_E &= \sum_{l_1+\cdots+l_s=l} \Gamma_{l_1}(x_1, \ldots, x_{l_1})\Gamma_{l_2}(x_1, \ldots, x_{l_2})\cdots\\
&\quad \times \Gamma_{l_s}(x_1, \ldots, x_{l_s}),
\end{aligned} \tag{A3}$$

where $l_i \geq 2$. Thus, the summation in the last equation is extended over all possible partitions of the set $x_1, x_2, \ldots, x_l$ into clusters of at least two points. In the case of a zero-mean Gaussian stochastic function, only the two-point correlation function $\Gamma_2(x_1, x_2)$ is non-vanishing; moments of $2l$, $\langle\gamma(x_1)\cdots\gamma(x_{2l})\rangle_E$, can be written as sums of $\frac{(2l)!}{2^l p!}$ terms, each of which is a product of two-point correlation functions; moments of odd order vanish.

(iii) The l-point correlation function has the following property (Frisch 1968): $\Gamma_l(x_1, x_2, \ldots, x_l)$ vanishes whenever the points $x_1, \ldots, l$ are not inside a common sphere whose diameter equals the correlation length a. Observe that the moment

$$\begin{aligned}
\langle\gamma(x_1)\gamma(x_2)\gamma(x_3)\gamma(x_4)\rangle_E &= \Gamma_2(x_1, x_2)\Gamma_2(x_3, x_4) + \Gamma_2(x_1, x_3)\Gamma_2(x_2, x_4)\\
&\quad + \Gamma_2(x_1, x_4)\Gamma_2(x_2, x_3) + \Gamma_4(x_1, x_2, x_3, x_4)
\end{aligned} \tag{A4}$$

does not satisfy this condition. If, for example,

$$|x_1 - x_2| < a; \quad |x_3 - x_4| < a; \quad |x_1 - x_3| \gg a, \tag{A5}$$

then

$$\langle\gamma(x_1)\gamma(x_2)\gamma(x_3)\gamma(x_4)\rangle_E = \Gamma_2(x_1, x_2)\Gamma_2(x_3, x_4) \neq 0, \tag{A6}$$

or if

$$|x_1 - x_3| < a; \quad |x_2 - x_4| < a; \quad |x_1 - x_2| \gg a, \tag{A7}$$

then

$$\langle\gamma(x_1)\gamma(x_2)\gamma(x_3)\gamma(x_4)\rangle_E = \Gamma_2(x_1, x_3)\Gamma_2(x_2, x_4) \neq 0. \tag{A8}$$

(iv) $\gamma(x)$ is assumed to be a spatially stationary and ergodic process.

We assume the above properties to hold at least locally.